A $(\log n)^{\Omega(1)}$ integrality gap for the Sparsest Cut SDP

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Abstract— We show that the Goemans-Linial semidefinite relaxation of the Sparsest Cut problem with general demands has integrality gap $(\log n)^{\Omega(1)}$. This is achieved by exhibiting n-point metric spaces of negative type whose L_1 distortion is $(\log n)^{\Omega(1)}$. Our result is based on quantitative bounds on the rate of degeneration of Lipschitz maps from the Heisenberg group to L_1 when restricted to cosets of the center.

Keywords-Sparsest Cut problem; semidefinite programming; integrality gap; metric embeddings; Heisenberg group.

1. Introduction

The L_1 distortion of a metric space (X, d), commonly denoted $c_1(X, d)$, is the infimum over $D > 0$ for which there exists a mapping $f : X \to L_1$ such that $\frac{\|f(x)-f(y)\|_1}{d(x,y)} \in [1, D]$ for all distinct $x, y \in X$. (If no such D exists we set $c_1(X, d) = \infty$). (X, d) is said to be a metric space of negative type, or a squared L_2 metric space, if the metric space $\left(X, \sqrt{d}\right)$ admits an isometric embedding into Hilbert space. A key example of a metric space of negative type is the Banach space L_1. The purpose of this paper is to prove the following result:

Theorem 1.1. *For every $n \in \mathbb{N}$ there exists an n-point metric space (X, d) of negative type such that*

$$c_1(X, d) \geq (\log n)^c,$$

where $c > 0$ is a universal constant which can be explicitly estimated (see Section 2).

The previous best known lower bound in the setting of Theorem 1.1 is $c_1(X, d) = \Omega(\log \log n)$: this is proved in [27] as an improved analysis of the spaces constructed in the breakthrough result of [25]. The best known upper bound [3] for the L_1 distortion of finite metric spaces of negative type is $c_1(X, d) = O\left((\log n)^{\frac{1}{2}+o(1)}\right)$, improving the previously known bounds of $O\left((\log n)^{\frac{3}{4}}\right)$ from [11] and the earlier bound of $O(\log n)$ from [7] which holds for arbitrary n-point metric spaces, i.e., without assuming negative type.

Next we discuss the significance of Theorem 1.1 in the context of approximation algorithms. The Sparsest Cut problem with general demands is a fundamental combinatorial optimization problem which is defined as follows. Given $n \in \mathbb{N}$ and two symmetric functions

$$C, D : \{1, \ldots, n\} \times \{1, \ldots, n\} \to [0, \infty)$$

(called capacities and demands, respectively) and a subset $S \subseteq \{1, \ldots, n\}$, write

$$\Phi(S) := \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} C(i, j) \cdot |\mathbf{1}_S(i) - \mathbf{1}_S(j)|}{\sum_{i=1}^{n} \sum_{j=1}^{n} D(i, j) \cdot |\mathbf{1}_S(i) - \mathbf{1}_S(j)|},$$

where $\mathbf{1}_S$ is the characteristic function of S. The value

$$\Phi^*(C, D) := \min_{S \subseteq \{1, \ldots, n\}} \Phi(S)$$

is the minimum over all cuts (partitions) of $\{1, \ldots, n\}$ of the ratio between the total capacity crossing the boundary of the cut and the total demand crossing the boundary of the cut.

Finding in polynomial time a cut for which $\Phi^*(C, D)$ is attained up to a definite multiplicative constant is called the Sparsest Cut problem, which is a basic step in approximation algorithms for several NP-hard problems [31], [1], [40], [10]. Computing $\Phi^*(C, D)$ exactly has been long-known to be NP-hard [39]. More recently, it was shown in [17] that there exists $\varepsilon_0 > 0$ such that it is NP-hard to approximate $\Phi^*(C, D)$ to within a factor smaller than $1 + \varepsilon_0$. In [25], [12] it was shown that it is Unique Games hard to approximate $\Phi^*(C, D)$ to within any constant factor (see [24] for more information on the Unique Games Conjecture).

The Sparsest Cut problem is the first algorithmic problem for which bi-Lipschitz embeddings of metric spaces were successfully used to design non-trivial polynomial time approximation algorithms [33], [6]. While early results were based on a remarkable approach using linear programming, an improved approach based on semidefinite programming (SDP) was put forth by Goemans and Linial in the late 1990s (see [23], [32]). This approach yields the best known approximation algorithm to the Sparsest Cut problem [3], which has an approximation guarantee of $O\left((\log n)^{\frac{1}{2}+o(1)}\right)$. The SDP approach of Goemans and Linial is based on

computing the following value:

$$M^*(C, D) := \min \left\{ \frac{\sum_{i=1}^n \sum_{j=1}^n C(i,j)d(i,j)}{\sum_{i=1}^n \sum_{j=1}^n D(i,j)d(i,j)} : \right.$$

$$\left. (\{1, \ldots, n\}, d) \text{ is a metric space of negative type} \right\}. \quad (1)$$

The minimization problem in (1) can be cast as a semidefinite program, and hence can be solved in polynomial time with arbitrarily good precision (see the explanation in [3]). It is also trivial to check that $M^*(C, D) \leq \Phi^*(C, D)$, i.e., (1) is a *relaxation* of the problem of computing $\Phi^*(C, D)$. The *integrality gap* of this SDP is the supremum of $\frac{\Phi^*(C,D)}{M^*(C,D)}$ over all symmetric functions $C, D : \{1, \ldots, n\} \times \{1, \ldots, n\} \to [0, \infty)$.

The integrality gap of the Goemans-Linial SDP is well known to equal the largest L_1 distortion of an n-point metric space of negative type in L_1. We recall the argument. The cut cone representation of L_1 metrics [19] states that a finite metric space (X, d) is isometric to a subset of L_1 if and only if it is possible to associate to every subset $S \subseteq X$ a nonnegative number $\lambda_S \geq 0$ such that the distance between any two points $x, y \in X$ can be computed via the formula:

$$d(x, y) = \sum_{S \subseteq X} \lambda_S |\mathbf{1}_S(x) - \mathbf{1}_S(y)|.$$

This fact immediately implies that for all symmetric functions $C, D : \{1, \ldots, n\} \times \{1, \ldots, n\} \to [0, \infty)$ we have:

$$\Phi^*(C, D) = \min_{x_1, \ldots, x_n \in L_1} \frac{\sum_{i=1}^n \sum_{j=1}^n C(i,j) \cdot \|x_i - x_j\|_1}{\sum_{i=1}^n \sum_{j=1}^n D(i,j) \cdot \|x_i - x_j\|_1}. \quad (2)$$

Thus, for all C, D and every metric d on $\{1, \ldots, n\}$ we have:

$$\frac{\sum_{i=1}^n \sum_{j=1}^n C(i,j)d(i,j)}{\sum_{i=1}^n \sum_{j=1}^n D(i,j)d(i,j)} \geq \frac{\Phi^*(C, D)}{c_1(\{1, \ldots, n\}, d)}. \quad (3)$$

Relation (2) and the bound (3) explain how the result of [3] quoted above yields an algorithm for Sparsest Cut with approximation guarantee of $O\left((\log n)^{\frac{1}{2} + o(1)}\right)$.

In the reverse direction, given any metric d on $\{1, \ldots, n\}$, by a duality argument (see Proposition 15.5.2 and Exercise 4 in chapter 15 of [34]) there exist symmetric functions $C_d, D_d : \{1, \ldots, n\} \times \{1, \ldots, n\} \to [0, \infty)$ (which arise in [34] from an appropriate separating hyperplane between certain convex cones) satisfying for every $x_1, \ldots, x_n \in L_1$:

$$\sum_{i=1}^n \sum_{j=1}^n C_d(i,j) \cdot \|x_i - x_j\|_1 \geq \sum_{i=1}^n \sum_{j=1}^n D_d(i,j) \cdot \|x_i - x_j\|_1, \quad (4)$$

and

$$\frac{\sum_{i=1}^n \sum_{j=1}^n C_d(i,j)d(i,j)}{\sum_{i=1}^n \sum_{j=1}^n D_d(i,j)d(i,j)} \leq \frac{1}{c_1(\{1, \ldots, n\}, d)}. \quad (5)$$

A combination of (2) and (4) shows that $\Phi^*(C_d, D_d) \geq 1$. Hence, choosing $C = C_d$ and $D = D_d$ in (3), together with (5), implies that actually $\Phi^*(C_d, D_d) = 1$ and:

$$\frac{\sum_{i=1}^n \sum_{j=1}^n C_d(i,j)d(i,j)}{\sum_{i=1}^n \sum_{j=1}^n D_d(i,j)d(i,j)} = \frac{\Phi^*(C_d, D_d)}{c_1(\{1, \ldots, n\}, d)}. \quad (6)$$

Substituting the metric d from Theorem 1.1 into (6) yields the following theorem:

Theorem 1.2. *For every $n \in \mathbb{N}$ there exist symmetric functions $C, D : \{1, \ldots, n\} \times \{1, \ldots, n\} \to [0, \infty)$ such that*

$$\frac{\Phi^*(C, D)}{M^*(C, D)} \geq (\log n)^c,$$

where $c > 0$ is the constant from Theorem 1.1. Thus, the integrality gap of the Goemans-Linial SDP for Sparsest Cut is $(\log n)^{\Omega(1)}$.

Remark 1.1. The Sparsest Cut problem has an important special case called the Uniform Sparsest Cut problem (or also Sparsest Cut with uniform demands). This problem corresponds to the case where $C(i, j) \in \{0, 1\}$ and $D(i, j) = 1$ for all $i, j \in \{1, \ldots, n\}$. In this case C induces a graph structure G on $V = \{1, \ldots, n\}$, where two distinct $i, j \in V$ are joined by an edge if and only if $C(i, j) = 1$. Thus for $S \subseteq V$ we have that $\Phi(S)$ is the number of edges joining S and $V \setminus S$ divided by $|S|(n - |S|)$, and hence $n\Phi^*(S)$ is, up to a factor of 2, the *edge expansion* of the graph G.

The best known approximation algorithm for the Uniform Sparsest Cut problem [4] achieves an approximation ratio of $O\left(\sqrt{\log n}\right)$, improving upon the previously best known bound [31] of $O(\log n)$. The $O\left(\sqrt{\log n}\right)$ approximation algorithm of [4] also uses the Goemans-Linial SDP relaxation described above. The best known lower bound [18] on the integrality gap of the Goemans-Linial SDP relaxation in the case of uniform demands is $\Omega(\log \log n)$.

Our integrality gap example in Theorem 1.1 works for the case of general demands, but *cannot* yield a lower bound tending to ∞ in the case of uniform demands, for the following reason. An inspection of the above argument shows that the integrality gap of the Goemans-Linial SDP in the case of uniform demands corresponds to the worst *average distortion* of negative type metrics d on $\{1, \ldots, n\}$ into L_1, i.e., the infimum over $D > 0$ such that for all negative type metrics d on $\{1, \ldots, n\}$ there exists a mapping $f : \{1, \ldots, n\} \to L_1$ for which

$$\|f(i) - f(j)\|_1 \leq Dd(x, y) \quad \forall i, j \in \{1, \ldots, n\},$$

and

$$\sum_{i,j=1}^n \|f(i) - f(j)\|_1 \geq \sum_{i,j=1}^n d(i, j).$$

This connection between the Uniform Sparsest Cut problem and average distortion embeddings is explained in detail in [38]. The metric spaces in Theorem 1.1 have doubling constant $O(1)$, and therefore by the proof in [38] they admit an embedding into the real line (and hence also into L_1) with

average distortion $O(1)$[1]. Thus our work does not provide progress on the problem of estimating the asymptotic behavior of the integrality gap of the SDP for Uniform Sparsest Cut, and it remains an interesting open problem to determine whether the currently best known lower bound, which is $\Omega(\log \log n)$, can be improved to $(\log n)^{\Omega(1)}$.

2. THE EXAMPLE

Define $\rho : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ by

$$
\begin{aligned}
&\rho\big((x, y, z), (t, u, v)\big) \\
&:= \left(\left[\left((t-x)^2 + (u-y)^2 \right)^2 + (v - z + 2xu - 2yt)^2 \right]^{\frac{1}{2}} \right. \\
&\qquad \left. + (t-x)^2 + (u-y)^2 \right)^{\frac{1}{2}}.
\end{aligned}
\tag{7}
$$

It was shown in [29] that $(\mathbb{R}^3, \rho)$ is a metric space of negative type. The result of [14] gives $c_1(\mathbb{R}^3, \rho) = \infty$, which implies that $c_1\big(\{0, 1, \ldots, k\}^3, \rho\big)$ tends to ∞ with k (The proof of this implication is via a compactness argument which would fail if $c_1(\mathbb{R}^3, \rho)$ were defined using the sequence space ℓ_1 rather than the function space L_1). Theorem 1.1 follows from a quantitative refinement of the statement $c_1(\mathbb{R}^3, \rho) = \infty$:

Theorem 2.1. *There exist universal constants $\psi, \delta > 0$ such that for all $k \in \mathbb{N}$ we have:*

$$
c_1\big(\{0, 1, \ldots, k\}^3, \rho\big) \geq \psi (\log k)^\delta.
$$

The proof of Theorem 2.1 is quite lengthy and involved. Complete details are given in the forthcoming full version of this paper [16]. Here we will give the key concepts and steps in the proof. First we wish to highlight a natural concrete open question that arises from Theorem 2.1. Denote:

$$
\delta^* := \limsup_{k \to \infty} \frac{\log \big(c_1\big(\{0, 1, \ldots, k\}^3, \rho\big) \big)}{\log \log k}.
$$

Combining the result of [3] and Theorem 2.1 shows that $\delta^* \in [\delta, 1/2]$ for some universal constant $\delta > 0$. In [16] we will give an explicit (though non-sharp) lower estimate on δ (just for the sake of stating a concrete bound in this paper, we can safely assert at this juncture that, say, $\delta \geq 2^{-1000}$). Proposition 7.10 in [16] (which we need to iterate 6 times) is the most involved step and essentially the only place in which sharpness has been sacrificed to simplify the exposition. We do not know how close an optimal version of our argument would come to yielding the constant δ^*. Conceivably $\delta^* = \frac{1}{2}$. If so, the metric spaces from Theorem 2.1 will already show that the integrality gap of the Sparsest Cut SDP is $\Theta\big((\log n)^{\frac{1}{2} + o(1)}\big)$.

3. QUANTITATIVE CENTRAL COLLAPSE

The main result of [14] states that if $U \subseteq \mathbb{R}^3$ is an open subset and if $f : U \to L_1$ is a Lipschitz function in the metric ρ defined in (7) then for almost every (with respect to Lebesgue measure) $(x, y, z) \in U$ we have

$$
\lim_{\varepsilon \to 0^+} \frac{\|f(x, y, z + \varepsilon) - f(x, y, z)\|_1}{\rho((x, y, z + \varepsilon), (x, y, z))} = 0.
\tag{8}
$$

Our main result is the following quantitative version of this statement:

Theorem 3.1. *There exists a universal constant $\delta \in (0, 1)$ with the following property. Let $B \subseteq \mathbb{R}^3$ be a unit ball in the metric ρ and let $f : B \to L_1$ be a function which is 1-Lipschitz with respect to ρ. Then for every $\varepsilon \in (0, 1/4)$ there exists $r \geq \varepsilon$ and $(x, y, z) \in B$ such that $(x, y, z + r) \in B$ and:*

$$
\frac{\|f(x, y, z + r) - f(x, y, z)\|_1}{\rho((x, y, z + r), (x, y, z))} \leq \frac{1}{(\log(1/\varepsilon))^\delta}.
$$

It was shown in Remark 1.6 of [29] that Theorem 3.1 (which was not known at the time) implies that if $X \subseteq \mathbb{R}^3$ is an η-net in the unit ball with respect to ρ centered at $(0, 0, 0)$ for some $\eta \in (0, 1/16)$ then $c_1(X, \rho) = \Omega(1)(\log(1/\eta))^\delta$. The key point of [29] is that one can use a Lipschitz extension theorem for doubling metric spaces [28] to extend an embedding of X into L_1 to a Lipschitz (but not bi-Lipschitz) function defined on all of $\mathbb{R}^3$ while incurring a universal multiplicative loss in the Lipschitz constant (in fact, since we are extending from a net, the existence of the required Lipschitz extension also follows from a simple partition of unity argument and there is no need to use the general result of [28]). Since the collapse result in Theorem 3.1 for this extended function occurs at a definite scale, one can use the fact that the function is bi-Lipschitz on the net X to obtain the required lower bound on the distortion. The metric space $\big(\frac{\{0,\ldots,k\}}{k} \times \frac{\{0,\ldots,k\}}{k} \times \frac{\{0,\ldots,k\}}{k^2}, \rho\big)$ is isometric to the metric space $\big(\{0, 1, \ldots, k\}^3, \frac{\rho}{k}\big)$, and it contains such an η net X with $\eta \approx \frac{1}{k}$. Hence Theorem 3.1 in conjunction with the above discussion implies Theorem 2.1.

In the remainder of this extended abstract we will explain the ingredients that go into the proof of Theorem 3.1.

4. THE HEISENBERG GROUP

Equip $\mathbb{R}^3$ with the following group structure:

$$
(a, b, c) \cdot (\alpha, \beta, \gamma) := (a + \alpha, b + \beta, c + \gamma + a\beta - b\alpha).
$$

The resulting non-commutative group is called the Heisenberg group, and is denoted $\mathbb{H}$. Note that the identity element of $\mathbb{H}$ is $e = (0, 0, 0)$ and the inverse of $(a, b, c) \in \mathbb{H}$ is $(-a, -b, -c)$. The center of $\mathbb{H}$ is $\{0\} \times \{0\} \times \mathbb{R}$. This explains why we call results such as (8) "central collapse".

For every $g = (a, b, c) \in \mathbb{H}$ we associate a special affine 2-plane, called the horizontal 2-plane at g, which is defined as $H_g = g \cdot (\mathbb{R}^2 \times \{0\})$. Thus H_e is simply the x, y plane. The

Carnot-Caratheodory metric on $\mathbb{H}$, denoted $d^{\mathbb{H}}$, is defined as follows: for $g, h \in \mathbb{H}$, $d^{\mathbb{H}}(g, h)$ is the infimum of lengths of smooth curves $\gamma : [0, 1] \to \mathbb{H}$ such that $\gamma(0) = g$, $\gamma(1) = h$ and for all $t \in [0, 1]$ we have $\gamma'(t) \in H_{\gamma(t)}$ (i.e., the tangent vector at time t is restricted to be in the corresponding horizontal 2-plane. The standard Euclidean norm on $\mathbb{R}^2$ induces a natural Euclidean norm on H_g for all $g \in \mathbb{H}$, and hence the norm of $\gamma'(t)$ is well defined for all $t \in [0, 1]$. This is how the length of γ is computed). For concreteness we mention that the metric $d^{\mathbb{H}}$ restricted to the integer grid $\mathbb{Z}^3$ is bi-Lipschitz equivalent to the word metric on $\mathbb{H}$ induced by the following (and hence any finite) canonical set of generators: $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ (in other words, this is simply the shortest path metric on the Cayley graph given by these generators). The metric space $(\mathbb{H}, d^{\mathbb{H}})$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, \rho)$ via the mapping $(x, y, z) \mapsto \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}, z \right)$ (this follows from the "ball-box theorem"—see for example [36]). Hence in what follows it will suffice to prove Theorem 3.1 with the metric ρ replaced by the metric $d^{\mathbb{H}}$.

Below, for $r > 0$ and $x \in \mathbb{H}$, we denote by $B_r(x)$ the open ball in the metric $d^{\mathbb{H}}$ of radius r centered at x. The following terminology will be used throughout this paper. A half space in $\mathbb{H}$ is the set of points lying on one side of some affine 2-plane in $\mathbb{R}^3$, including the points of the plane itself. A half space is called horizontal if its associated 2-plane is of the form H_g for some $g \in \mathbb{H}$. Otherwise the half space is called vertical. An affine line in $\mathbb{R}^3$ which passes through some point $g \in \mathbb{H}$ and lies in the plane H_g is called a horizontal line. The set of all horizontal lines in $\mathbb{H}$ is denoted lines($\mathbb{H}$).

5. Cut measures and sets of finite perimeter

In what follows we set $B = B_1(e) = B_1((0, 0, 0))$ and fix a 1-Lipschitz function $f : B \to L_1$ (in the metric $d^{\mathbb{H}}$). The cut (semi)-metric associated to a subset $E \subseteq B$ is defined as $d_E(x, y) := |\mathbf{1}_E(x) - \mathbf{1}_E(y)|$. Let Cut($B$) denote the space of all measurable cuts (subsets) of B equipped with the semi-metric given by the Lebesgue measure of the symmetric difference. In [14] a measure theoretic version of the cut-cone representation was studied. It states that there is a canonical Borel measure Σ_f on Cut(B) such that for almost all $x, y \in B$ we have:

$$d_f(x, y) := \| f(x) - f(y) \|_1 = \int_{\text{Cut}(B)} d_E(x, y) d\Sigma_f(E). \quad (9)$$

A key new ingredient of the result of [14] is that the Lipschitz condition on f forces the measure Σ_f to be supported on cuts with additional structure, namely cuts with finite perimeter. For sets with smooth boundary the perimeter is a certain explicit integral with respect to the surface area measure on the boundary (and, in the case of $\mathbb{R}^3$ equipped with the Euclidean metric, it simply coincides with the surface area for smooth sets). However, since the sets appearing in the representation (9) cannot be a priori

enforced to have any smoothness properties we need to work with a measure theoretical extension of the notion of surface area. Namely, define for every $E \in \text{Cut}(B)$, and an open set $U \subseteq \mathbb{H}$

$$\text{Per}(E)(U) := \inf \left\{ \liminf_{i \to \infty} \int_U \text{Lip}_x(h_i) d\mu(x) : \{h_i\}_{i=1}^{\infty} \right.$$

$$\left. \text{Lipschitz functions tending to } \mathbf{1}_E \text{ in } L_1^{\text{loc}}(B) \right\}. \quad (10)$$

Here, and in what follows, μ denotes the Lebesgue measure on $\mathbb{H} = \mathbb{R}^3$ and for $h : \mathbb{H} \to \mathbb{R}$ the quantity

$$\text{Lip}_x(h) := \limsup_{y \to x} \frac{|h(y) - h(x)|}{d^{\mathbb{H}}(x, y)}$$

denotes the local Lipschitz constant of h at x. Convergence in $L_1^{\text{loc}}(B)$ means, as usual, convergence in $L_1(K, \mu)$ for all compact subsets $K \subseteq B$. (To get some intuition for this notion, consider the analogous definition in the Euclidean space $\mathbb{R}^3$, i.e., when the functions $\{h_i\}_{i=1}^{\infty}$ are assumed to be Lipschitz with respect to the Euclidean metric rather than the metric $d^{\mathbb{H}}$. In this case, for sets E with smooth boundary, the quantity $\text{Per}(E)(U)$ is the surface area of the part of the boundary of E which is contained in U). $\text{Per}(E)(\cdot)$ can be extended to be a Radon measure on $\mathbb{H}$ (see for example [2]). A key insight of [14] is that the fact that f is 1-Lipschitz implies that for every open subset $U \subseteq B$ we have:

$$\int_{\text{Cut}(B)} \text{Per}(E)(U) d\Sigma_f(E) \leq C \cdot \mu(U), \quad (11)$$

where C is a universal constant (independent of f). Also there is an induced total perimeter measure λ_f defined by:

$$\lambda_f(\cdot) := \int_{\text{Cut}(B)} \text{Per}(E)(\cdot) d\Sigma_f(E). \quad (12)$$

In [14] the inequality (11) was used to show that $\mathbb{H}$ does not admit a bi-Lipschitz embedding into L_1 by exploiting the infinitesimal regularity of sets of finite perimeter. Specifically, let $E \subseteq \mathbb{H}$ be a set with finite perimeter. Then, as proved in [20], [21], with respect to the measure $\text{Per}(E)$, for almost every $p \in E$, asymptotically under blow up the measure of the symmetric difference of E and some unique *vertical* half space goes to 0. Intuitively, this means that (in a measure theoretic sense) almost every point $p \in \partial E$ has a tangent 2-plane which is vertical. Observe that a cut semi-metric associated to a vertical half-space, when restricted to a coset of the center of $\mathbb{H}$, is identically 0. This fact together with (9) suggests that under blow-up, at almost all points, f becomes degenerate in the direction of cosets of the center, and therefore $\mathbb{H}$ does not admit a bi-Lipschitz embedding into L_1. This is the heuristic argument behind the main result of [14]. What is actually required is a version of the results of [20], [21] for measured families of finite perimeter cuts corresponding to the representation (9).

The verticality, which played a key role above, is an initially surprising feature of the Heisenberg geometry, which in actuality, can easily be made intuitively plausible. We will not do so here since below we do not use it. What we do use is a quantitative version of a cruder statement, which in effect ignores the issues of verticality and uniqueness of generalized tangent planes. This suffices for our purposes. Our approach incorporates ideas from a second and simpler proof of the (non-quantitative) bi-Lipschitz non-embeddability of $\mathbb{H}$ into L_1, which was obtained in [15]. The second proof, which did not require the results of [20], [21], is based on the notion of *monotone sets* which we now describe.

6. Monotone sets

Fix an open set $U \subseteq \mathbb{H}$. Let lines(U) denote the space of unparametrized oriented horizontal lines whose intersection with U is nonempty. Let $\mathcal{N}_U$ denote the unique left invariant measure on lines$(\mathbb{H})$ normalized so that $\mathcal{N}_U(\text{lines}(U)) = 1$. A subset $E \subseteq U$ is *monotone with respect to U* if for $\mathcal{N}_U$-almost every line L, both $E \cap L$ and $(U \setminus E) \cap L$ are essentially connected, in the sense that there exist connected subsets $F_L = F_L(E), F'_L = F'_L(E) \subseteq L$ (i.e., each of F_L, F'_L is either empty, equals L, or is an interval, or a ray in L) such that the symmetric differences $(E \cap L) \triangle F_L$ and $((U \setminus E) \cap L) \triangle F'_L$ have 1-dimensional Hausdorff measure 0.

When $U = \mathbb{H}$, a non-trivial classification theorem was proved in [15], stating that if E is monotone with respect to $\mathbb{H}$ then either E or $\mathbb{H} \setminus E$ has measure zero, or there exists a half space $\mathcal{P}$ such that $\mu(E \triangle \mathcal{P}) = 0$. Note for the sake of comparison with the Euclidean case that if we drop the requirement that the lines are horizontal in the definition of monotone sets then monotonicity would essentially mean that (up to sets of measure 0) both E and the complement of E are convex sets, and hence E is a half space up to a set of measure 0. The non-trivial point in the classification result of [15] is that we are allowed to work only with a codimension 1 subset of all affine lines in $\mathbb{R}^3$, namely the horizontal lines.

Using the above classification result for monotone sets, in [15] the non-embedding result for $\mathbb{H}$ in L_1 is proved by using once more a blow-up argument (or metric differentiation) to reduce the non-embedding theorem to the special case in which the cut measure Σ_f is supported on sets which are monotone with respect to $\mathbb{H}$. Thus, the cut measure is actually supported on half spaces. It follows (after the fact) that the connectedness condition in the definition of monotone sets holds for every line L, not just for horizontal lines. This implies that for *every affine* line L, if $x_1, x_2, x_3 \in L$ and x_2 lies between x_1 and x_3 then

$$\|f(x_1) - f(x_3)\|_1 = \|f(x_1) - f(x_2)\|_1 + \|f(x_2) - f(x_3)\|_1. \quad (13)$$

But if L is vertical then $d^{\mathbb{H}}|_L$ is bi-Lipschitz to the *square root* of the difference of the z-coordinates, and it is trivial

to verify that this metric on L is not bi-Lipschitz equivalent to a metric on L satisfying (13).

In proving Theorem 3.1, the most difficult part by far is a stability theorem stating in quantitative form that individual cuts which are "approximately monotone" are close to half spaces; see Theorem 7.1. Here, it is important to have the right notion of "approximately monotone". We also show that on a controlled scale, modulo a controlled error, we can at most locations reduce to the case when the cut measure is supported on cuts which are approximately close to being monotone so that Theorem 7.1 can be applied, and such that in addition there is a bound on the total cut measure. For this, the bound (11) is crucially used to estimate the scale at which the "total non-monotonicity" is appropriately small. At such a good scale and location, it now follows that up to a small controlled error (13) holds. In the next section we introduce the notion of δ-monotone sets and state the stability theorem which ensures that δ-monotone sets are close to half spaces on a ball of controlled size.

7. Stability of monotone sets

Denote $\mathcal{N} = \mathcal{N}_B$, i.e., $\mathcal{N}$ is the left invariant measure on lines$(\mathbb{H})$ normalized so that the measure of the horizontal lines that intersect B is 1. For a horizontal line $L \in \text{lines}(\mathbb{H})$ let $\mathcal{H}_L^1$ denote the 1-dimensional Hausdorff measure on L with respect to the metric induced from $d^{\mathbb{H}}$.

Fix a ball $B_r(x) \subseteq B$. For every measurable $E \subseteq \mathbb{H}$ and $L \in \text{lines}(B_r(x))$ we define the non-convexity of (E, L) on $B_r(x)$ by:

$$\text{NC}_{B_r(x)}(E, L) := \inf \left\{ \int_{L \cap B_r(x)} \left| \mathbf{1}_I - \mathbf{1}_{E \cap L \cap B_r(x)} \right| d\mathcal{H}_L^1 : \right.$$

$$\left. I \subseteq L \cap B_r(x) \text{ subinterval} \right\}. \quad (14)$$

The non-monotonicity of (E, L) on $B_r(x)$ is defined as:

$$\text{NM}_{B_r(x)}(E, L) := \text{NC}_{B_r(x)}(E, L) + \text{NC}_{B_r(x)}(\mathbb{H} \setminus E, L).$$

The non-monotonicity of E on $B_r(x)$ is defined as:

$$\text{NM}_{B_r(x)}(E) := \frac{1}{r^4} \int_{\text{lines}(B_r(x))} \text{NM}_{B_r(x)}(E, L) d\mathcal{N}(L)$$

$$= \frac{1}{\mathcal{N}(\text{lines}(B_r(x)))} \int_{\text{lines}(B_r(x))} \frac{\text{NM}_{B_r(x)}(E, L)}{r} d\mathcal{N}(L).$$

Note that by design $\text{NM}_{B_r(x)}(E)$ is a scale invariant quantity. A measurable set E is said to be δ-monotone on $B_r(x)$ if $\text{NM}_{B_r(x)}(E) < \delta$. Our stability result for monotone sets is the following theorem:

Theorem 7.1. *There exists a universal constant $a > 0$ such that if a measurable set $E \subseteq B_r(x)$ is ε^a-monotone on $B_r(x)$ then there exists a half-space $\mathcal{P}$ such that*

$$\frac{\mu((E \cap B_{\varepsilon r}(x)) \triangle \mathcal{P})}{\mu(B_{\varepsilon r}(x))} < \varepsilon^{1/3}.$$

The proof of Theorem 7.1 constitutes the bulk of the full version of this paper [16]. Formally, it follows the steps of the argument of [15] in the case of sets which are precisely monotone. However, substantial additions are required arising from the need to work with certain appropriate quantitatively defined notions of "fuzzy" measure theoretical boundaries of sets, and by the need to make a certain existence statement of [15] quantitative.

8. Splitting the cut measure

Theorem 7.1 will allow us to control individual integrands in the cut representation (9) (assuming that we can find a scale at which the total non-monotonicity is small enough—this is discussed in Section 9 below). But, such point-wise estimates do not suffice since we do not have any a priori control on the total mass of the cut measure Σ_f. To overcome this problem we split the measure Σ_f into two parts in such a way that one part has controlled total mass, while the other part contributes a negligible amount to the metric d_f.

Fix a ball $B_r(p) \subseteq B$. In what follows we will use the notation $\lesssim, \gtrsim$ to denote the corresponding inequalities up to universal factors. We shall also use the fact that $\mu(B_s(z)) = s^4 \mu(B_1(z))$ for all $s > 0$ and $z \in \mathbb{H}$.

For $\theta > 0$ define $D_\theta \subseteq \mathrm{Cut}(B)$ by

$$D_\theta := \{E \in \mathrm{Cut}(B) : \ \mathrm{Per}(E)(B_r(p)) > \theta\mu(B_r(p))\}.$$

Markov's inequality combined with (11) implies that $\Sigma_f(D_\theta) \lesssim \frac{1}{\theta}$. Define a semi-metric d_θ on $\mathbb{H}$ by

$$d_\theta(x, y) := \int_{D_\theta} d_E(x, y) d\Sigma_f(E).$$

We claim that even though we do not have a bound on $\Sigma_f(\mathrm{Cut}(B))$ we can still control the distance between d_f and d_θ in $L_1(B_r(p) \times B_r(p))$. This can be deduced from the isoperimetric inequality on $\mathbb{H}$ (see [9]) which implies that for every $E \in \mathrm{Cut}(B)$ we have

$$\frac{\mu(B_r(p) \cap E)}{\mu(B_r(p))} \cdot \frac{\mu(B_r(p) \setminus E)}{\mu(B_r(p))} \lesssim \left(\frac{r}{\mu(B_r(p))} \mathrm{Per}(E)(B_r(p))\right)^{4/3},$$

or

$$\mu(B_r(p) \cap E)\mu(B_r(p) \setminus E) \lesssim r^4 \left(\mathrm{Per}(E)(B_r(p))\right)^{4/3}. \quad (15)$$

The argument is as follows: for each non-negative integer n define

$$A_n := \left\{E \in \mathrm{Cut}(B) : \right.$$
$$\left. \frac{\theta\mu(B_r(p))}{2^{n+1}} < \mathrm{Per}(E)(B_r(p)) \leq \frac{\theta\mu(B_r(p))}{2^n}\right\}.$$

Then

$$\mathrm{Cut}(B) \setminus D_\theta = \left(\bigcup_{n=0}^{\infty} A_n\right) \bigcup A_\infty,$$

where

$$A_\infty := \{E \in \mathrm{Cut}(B) : \ \mathrm{Per}(E)(B_r(p)) = 0\}.$$

Markov's inequality combined with (11) implies that $\Sigma_f(A_n) \lesssim \frac{2^n}{\theta}$ for all n, while (15) implies that for each $E \in A_n$ we have

$$\mu(B_r(p) \cap E)\mu(B_r(p) \setminus E) \lesssim r^{28/3} \left(\frac{\theta}{2^n}\right)^{4/3}$$

and for $E \in A_\infty$ we have $\mu(B_r(p) \cap E)\mu(B_r(p) \setminus E) = 0$. We therefore obtain the estimate:

$$\|d_f - d_\theta\|_{L_1(B_r(p) \times B_r(p))} = \int_{B_r(p) \times B_r(p)}$$
$$\left(\int_{\mathrm{Cut}(B_r(p)) \setminus D_\theta} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| d\Sigma_f(E)\right) d\mu(x)d\mu(y)$$
$$= \sum_{n=0}^{\infty} \int_{A_n} 2\mu(B_r(p) \cap E)\mu(B_r(p) \setminus E) d\Sigma_f(E)$$
$$\lesssim \sum_{n=1}^{\infty} \frac{2^n}{\theta} r^{28/3} \left(\frac{\theta}{2^n}\right)^{4/3} \lesssim r^{28/3}\theta^{1/3}. \quad (16)$$

9. Controlling the scale at which the total non-monotonicity is small

We shall require a formula, known as a kinematic formula, which expresses the perimeter of a set $E \subseteq \mathbb{H}$ as an integral over the space of lines L of the perimeter of the 1-dimensional sets $E \cap L$. This formula (proved in Proposition 3.13 of [35]) asserts that there exists a constant $\gamma = \gamma(\mathbb{H})$ such that for every open subset $U \subseteq \mathbb{H}$ and a measurable subset $E \subseteq \mathbb{H}$ with $\mathrm{Per}(E)(U) < \infty$ the function $L \mapsto \mathrm{Per}(E \cap L)(U \cap L)$ from $\mathrm{lines}(U)$ to $[0, \infty)$ is in $L_1(\mathrm{lines}(U), \mathcal{N})$ and satisfies the identity:

$$\mathrm{Per}(E)(U) = \gamma \int_{\mathrm{lines}(U)} \mathrm{Per}(E \cap L)(U \cap L) d\mathcal{N}(L). \quad (17)$$

Here we used the notion of one dimensional perimeter, which is defined analogously to (10). For one dimensional sets with finite perimeter the notion of perimeter has a simple characterization (see Proposition 3.52 in [2]). Whenever $\mathrm{Per}(E \cap L)(U \cap L) < \infty$ there exists a unique collection of finitely many disjoint intervals

$$\mathcal{I}(E, L, U) = \{I_1(E, L, U), \ldots, I_n(E, L, U)\}$$

which are relatively closed in $L \cap U$ and such that the symmetric difference of $E \cap L$ and $\bigcup_{j=1}^{n} I_j(E, L, U)$ has measure 0. The perimeter measure $\mathrm{Per}(E \cap L)$ is the sum of delta functions concentrated at the end points of these intervals and hence $\mathrm{Per}(E \cap L)(U \cap L)$ is the number of these end points.

Fix $\delta \in (0, 1/2)$. For every non-negative integer j let $C_j(E, L)$ denote the collection of intervals in $\mathcal{I}(E, L, B)$ whose length is in $\left(2\delta^{j+1}, 2\delta^j\right]$. Let $\mathcal{E}_j(E, L)$ denote the

collection of all end points of intervals in $C_j(E, L)$. For a measurable $A \subseteq B$ write:

$$w_j(E)(A) :=$$
$$\gamma \int_{\text{lines}(B)} \frac{|A \cap \mathcal{E}_j(E, L)| + |A \cap \mathcal{E}_j(\mathbb{H} \setminus E, L)|}{2} d\mathcal{N}(L), \quad (18)$$

where γ is as in (17). We also set:

$$w_j(A) = \int_{\text{Cut}(B)} w_j(E)(A) d\Sigma_f(E). \quad (19)$$

The kinematic formula (17) implies that

$$\lambda_f = \sum_{j=0}^{\infty} w_j. \quad (20)$$

It follows from (20) that

$$\sum_{j=0}^{\infty} w_j(B) = \lambda_f(B) \lesssim 1. \quad (21)$$

Thus there exists $j \le \delta^{-1}$ for which $w_j(B) \lesssim \delta$. We shall fix this integer j from now on. The ball B contains $\gtrsim \delta^{-4j}$ disjoint balls of radius δ^j. Thus there exists $y \in B$ such that $B_{\delta^j}(y) \subseteq B$ and $w_j(B_{\delta^j}(y)) \lesssim \delta^{4j+1}$. We shall fix this point $y \in B$ from now on.

Fix $E \subseteq \mathbb{H}$ with finite perimeter. For $\mathcal{N}$-almost every $L \in \text{lines}\left(B_{\delta^j(y)}\right)$ the set $\mathcal{I}(E, L, B_{\delta^j}(y))$ consists of finitely many intervals $I_1, \ldots, I_n$. Note that each of the intervals $I_1, \ldots, I_n$ (including both endpoints) is contained in the closure of $B_{\delta^j}(y)$, and hence its length is at most $2\delta^j$. It follows that each of these intervals lies in $C_k(E, L)$ for some $k \ge j$. By the definition (14) we have:

$$\text{NC}_{B_{\delta^j}(y)}(E, L) \le \sum_{s=1}^{n} \text{length}(I_s) \lesssim \sum_{k \ge j} \delta^k |B_{\delta^j}(y) \cap \mathcal{E}_k(E, L)|.$$

Arguing similarly for $\mathbb{H} \setminus E$ yields:

$$\text{NM}_{B_{\delta^j}(y)}(E, L) \lesssim$$
$$\sum_{k \ge j} \delta^k \left(|B_{\delta^j}(y) \cap \mathcal{E}_k(E, L)| + |B_{\delta^j}(y) \cap \mathcal{E}_k(\mathbb{H} \setminus E, L)|\right). \quad (22)$$

Averaging (22) over $L \in \text{lines}(B_{\delta^j}(y))$ gives a bound on the total non-monotonicity:

$$\text{NM}_{B_{\delta^j}(y)}(E) \lesssim \delta^{-4j} \sum_{k \ge j} \delta^k \int_{\text{lines}\left(B_{\delta^j}(y)\right)}$$
$$\left(|B_{\delta^j}(y) \cap \mathcal{E}_k(E, L)| + |B_{\delta^j}(y) \cap \mathcal{E}_k(\mathbb{H} \setminus E, L)|\right) d\mathcal{N}(L)$$
$$\overset{(18)}{\lesssim} \delta^{-4j} \sum_{k \ge j} \delta^k w_k(E)(B_{\delta^j}(y)). \quad (23)$$

Integrating (23) with respect to $E \in \text{Cut}(B)$ and using (19) yields the bound:

$$\int_{\text{Cut}(B)} \text{NM}_{B_{\delta^j}(y)}(E) d\Sigma_f(E) \lesssim \delta^{-4j} \sum_{k \ge j} \delta^k w_k(B_{\delta^j}(y))$$
$$\overset{(20)}{\le} \delta^{-3j} w_j(B_{\delta^j}(y)) + \delta^{-4j} \cdot \delta^{j+1} \lambda_f(B_{\delta^j}(y))$$
$$\overset{(11)}{\lesssim} \delta^{-3j} w_j(B_{\delta^j}(y)) + \delta^{-3j+1} \mu(B_{\delta^j}(y))$$
$$\lesssim \delta^{-3j} w_j(B_{\delta^j}(y)) + \delta^{j+1} \lesssim \delta^{j+1}, \quad (24)$$

where in the last inequality above we used our choice of y and j which ensures that $w_j(B_{\delta^j}(y)) \lesssim \delta^{4j+1}$.

10. Cut metrics close to ones supported on almost half spaces

Let $\Sigma_{\mathcal{P}}$ be a measure on $\text{Cut}(B)$ which is supported on half spaces. Assume that

$$\|d_{\mathcal{P}} - d_f\|_{L_1(B \times B)} \le \varepsilon.$$

Our goal is to use this assumption to deduce that d_f must collapse some pair of points lying on the same coset of the center whose distance is controlled from below by an appropriate power of ε. Namely, we will show that there exist $x, y \in B$ lying on the same coset of the center such that

$$d^{\mathbb{H}}(x, y) \gtrsim \varepsilon^{1/90} \quad \text{yet} \quad d_f(x, y) \lesssim \varepsilon^{1/18} d^{\mathbb{H}}(x, y).$$

This step is a quantitative (integral) version of the argument that was sketched in Section 6, which relies on the fact that $d_{\mathcal{P}}$ is additive along every affine line.

Define $u = (0, 0, h)$ where $h > 0$ is a small enough universal constant such that $\frac{1}{4} \le d^{\mathbb{H}}(u, e) \le \frac{1}{2}$. Consider the set $A \subseteq B \times B$ consisting of pairs of points which lie on a line segment joining a point $p \in B_{\varepsilon^{1/9}}(e)$ and a point $q \in B_{\varepsilon^{1/9}}(u)$. Then $\mu \times \mu(A) \gtrsim \varepsilon^{8/9}$, so that our assumption implies that

$$\frac{1}{\mu \times \mu(A)} \|d_{\mathcal{P}} - d_f\|_{L_1(A)} \le \varepsilon^{1/9}.$$

By a Fubini argument it follows that there exist $p \in B_{\varepsilon^{1/9}}(e)$ and $q \in B_{\varepsilon^{1/9}}(u)$ such that if we denote by $I = [p, q]$ the line segment joining p and q then

$$\|d_{\mathcal{P}} - d_f\|_{L_1(I \times I)} \lesssim \varepsilon^{1/9}.$$

Fix an integer $n \approx \varepsilon^{-1/45}$. For $i \in \{0, \ldots, n\}$ let

$$J_i = \left[\frac{2i}{2n+1}, \frac{2i+1}{2n+1}\right] \subseteq [0, 1].$$

Then for every $(t_0, \ldots, t_n) \in J_0 \times \cdots \times J_n$ the additivity of $d_{\mathcal{P}}$ on the line segment I implies that

$$d_P(v_{t_n}, v_{t_0}) = \sum_{i=0}^{n-1} d_{\mathcal{P}}(v_{t_i}, v_{t_{i+1}}).$$

Integrating this equality over $J_0 \times \cdots \times J_n$ we get

$$\int_{J_n \times J_0} d_{\mathcal{P}}(v_s, v_t) \, ds dt = \sum_{i=0}^{n-1} \int_{J_i \times J_{i+1}} d_{\mathcal{P}}(v_s, v_t) \, ds dt.$$

Since $\|d_{\mathcal{P}} - d_f\|_{L_1(I \times I)} \lesssim \varepsilon^{1/9}$ it follows that

$$\sum_{i=0}^{n-1} \int_{J_i \times J_{i+1}} d_f(v_s, v_t) \, ds dt \lesssim \int_{J_n \times J_0} d_f(v_s, v_t) \, ds dt + n\varepsilon^{1/9}.$$

Assume that for all $i \in \{0, \ldots, n-1\}$ and $(v_s, v_t) \in J_i \times J_{i+1}$ we have $d_f(v_s, v_t) \geq \frac{\beta}{\sqrt{n}}$. Then using the fact that f is 1-Lipschitz we arrive at the bound $n \cdot \frac{1}{n^2} \frac{\beta}{\sqrt{n}} \lesssim \frac{1}{n^2} + n\varepsilon^{1/9}$, and therefore $\beta \lesssim n^{5/2}\varepsilon^{1/9} \lesssim \varepsilon^{1/18}$.

We proved above that there exists $i \in \{0, \ldots, n-1\}$ and $(v_s, v_t) \in J_i \times J_{i+1}$ such that

$$d_f(v_s, v_t) \lesssim \frac{\varepsilon^{1/18}}{\sqrt{n}}.$$

Writing $v_s = (a_1, a_2, a_3)$ and $v_t = (b_1, b_2, b_3)$ one checks that $|a_1 - b_1|, |a_2 - b_2| \lesssim \varepsilon^{1/9}$ and $|a_3 - b_3| \approx \frac{1}{\sqrt{n}}$. Therefore if we set $w = (a_1, a_2, b_3)$ then v_s and w lie on the same coset of the center and $d^{\mathbb{H}}(v_s, w) \approx \frac{1}{\sqrt{n}} \approx \varepsilon^{1/90}$ while

$$d_f(v_s, w) \lesssim d_f(v_s, v_t) + d_f(v_t, w) \lesssim \frac{\varepsilon^{1/18}}{\sqrt{n}} + \varepsilon^{1/9}$$
$$\lesssim \varepsilon^{1/18} d^{\mathbb{H}}(v_s, w),$$

as required.

11. Putting things together

Fix $\varepsilon > 0$ and take $\delta = \varepsilon^K$ for a large enough $K > a$ that will be determined presently, where a is as in Theorem 3.1. Let j and y be as in Section 9 for this value of δ, i.e., (24) is satisfied. Thus $j \leq \varepsilon^{-K}$. We now define

$$M := \left\{ E \in \mathrm{Cut}(B) : \ \mathrm{NM}_{B_{\delta j}(y)}(E) \leq \varepsilon^a \right\}.$$

Then by Markov's inequality applied to (24) we are ensured that

$$\Sigma_f(\mathrm{Cut}(B) \setminus M) \lesssim \varepsilon^{K-a}\delta^j.$$

Define two semi-metrics on B by

$$d_1(p, q) := \int_M d_E(p, q) d\Sigma_f(E)$$

and

$$d_2(p, q) := \int_{\mathrm{Cut}(B) \setminus M} d_E(p, q) d\Sigma_f(E) = d_f - d_1.$$

Then for all $p, q \in B_{\delta j}(y)$ we have $d_2(p, q) \lesssim \varepsilon^{K-a}\delta^j$.

By the definition of M, for all $E \in M$ Theorem 7.1 implies that there exists a half space $\mathcal{P}_E$ for which

$$\mu \left((E \cap B_{\varepsilon\delta j}(x)) \triangle \mathcal{P}_E \right) \lesssim \varepsilon^{1/3} \left(\varepsilon\delta^j \right)^4. \tag{25}$$

We shall now use the splitting of the cut measure from Section 8 with $r := \varepsilon\delta^j$, $p = y$, and a parameter $\theta > 0$ which

will be determined presently. Define two semi-metrics on B by

$$d_3(u, v) := \int_{M \cap D_\theta} d_E(u, v) d\Sigma_f(E)$$

and

$$\rho(u, v) := \int_{M \cap D_\theta} d_{\mathcal{P}_E}(u, v) d\Sigma_f(E)$$

(here $D_\theta \subseteq \mathrm{Cut}(B)$ is as in Section 8). Then

$$\|d_3 - \rho\|_{L_1(B_r(y) \times B_r(y))} \leq \int_{M \cap D_\theta}$$
$$\left(\int_{B_r(y) \times B_r(y)} \left| |\mathbf{1}_E(u) - \mathbf{1}_{\mathcal{P}_E}(u)| + |\mathbf{1}_E(v) - \mathbf{1}_{\mathcal{P}_E}(v)| \right| \right) d\Sigma_f(E)$$
$$\overset{(25)}{\lesssim} \Sigma_f(D_\theta)\varepsilon^{1/3} r^8 \lesssim \frac{\varepsilon^{1/3} r^8}{\theta}, \tag{26}$$

where in the last inequality of (26) we used the bound $\Sigma_f(D_\theta) \lesssim \frac{1}{\theta}$ from Section 8. Note that with d_θ as in Section 8 we have the point-wise inequality

$$|d_\theta - d_3| \leq d_2 \leq \varepsilon^{K-a}\delta^j = \varepsilon^{K-a-1}r.$$

Now,

$$\frac{\|d_f - \rho\|_{L_1(B_r(y) \times B_r(y))}}{\mu(B_r(y))^2}$$
$$\lesssim \frac{\|d_f - d_\theta\|_{L_1(B_r(y) \times B_r(y))}}{r^8} + \frac{\|d_\theta - d_3\|_{L_1(B_r(y) \times B_r(y))}}{r^8}$$
$$+ \frac{\|d_3 - \rho\|_{L_1(B_r(y) \times B_r(y))}}{r^8}$$
$$\overset{(16)\wedge(26)}{\lesssim} \frac{r^{28/3}\theta^{1/3} + \varepsilon^{K-a-1} r \cdot r^8 + \varepsilon^{1/3} r^8 \theta^{-1}}{r^8}$$
$$= r^{4/3}\theta^{1/3} + \frac{\varepsilon^{1/3}}{\theta} + \varepsilon^{K-a-1}r. \tag{27}$$

The optimal choice of θ in (27) is $\theta \approx \frac{\varepsilon^{1/4}}{r}$. This yields the bound

$$\frac{\|d_f - \rho\|_{L_1(B_r(y) \times B_r(y))}}{\mu(B_r(y))^2} \lesssim r \left(\varepsilon^{1/12} + \varepsilon^{K-a-1} \right) \lesssim \varepsilon^{1/12} r,$$

provided that $K - a - 1 \geq \frac{1}{12}$. The result of Section 10 now implies that there exist $w, z \in B_r(y)$ which lie on the same coset of the center and

$$d^{\mathbb{H}}(w, z) \gtrsim \varepsilon^{1/1080} r \quad \text{yet} \quad d_f(w, z) \lesssim \varepsilon^{1/216} d^{\mathbb{H}}(w, z).$$

Since $j \leq \frac{1}{\delta}$ and $\delta = \varepsilon^K$ we see that

$$d^{\mathbb{H}}(w, z) \gtrsim \varepsilon^{1/1080} \cdot \varepsilon\delta^j \geq \varepsilon^{2+K\varepsilon^{-K}} \geq e^{-\varepsilon^{-2K}}$$

for ε small enough. The proof of Theorem 3.1 is complete.

12. Concluding remarks

We have presented here the complete details of the proof of Theorem 1.1, assuming only Theorem 7.1 on the stability of monotone sets, whose proof constitutes the bulk of [16]. The obvious significance of Theorem 1.1 is that it shows that the correct asymptotic "ballpark" of the integrality gap of the Sparsest Cut SDP is in the power of $\log n$ range. But, this result has other important features, the most notable of which is that it shows that the L_1 distortion of doubling, and hence also decomposable, n-point metric spaces can grow like $(\log n)^{\Omega(1)}$ (we refer to [29] for an explanation of the significance of this statement). Moreover, unlike the construction of [25] which was tailored especially for this problem, the Heisenberg group is a classical and well understood object, which in a certain sense (which can be made precise), is the smallest possible L_1 non-embeddable metric space of negative type which posses certain symmetries (an invariant metric on a group that behaves well under dilations).

In addition to the above discussion, our proof contains several ideas and concepts which are of independent interest and might be useful elsewhere. Indeed, the monotonicity and metric differentiation approach to L_1-valued Lipschitz maps, as announced (and sketched) in Section 1.8 of [14], was also used in a much simpler form in [30], in a combinatorial context and for a different purpose. Our proof is in a sense a "hybrid" argument, which uses ideas from [14], as well as the simplified proof in [15], with a crucial additional ingredient to estimate the scale. We prove a stability version of the classification of monotone sets in [15], but unlike [15] we also need to work with perimeter bounds following [14] in order to deal with (using the isoperimetric inequality on $\mathbb{H}$ as in Section 8) the issue that the total mass of the cut measure does not have an a priori bound. In addition, the bound on the total perimeter is shown via the kinematic formula to lead to a bound on the total non-monotonicity, which in turn, leads to the scale estimate.

It is often the case in combinatorics and theoretical computer science that arguments which are most natural to discover and prove in the continuous domain need to be discretized. The "vanilla approach" to such a discretization would be to follow the steps of the proof of the continuous/analytic theorem on the corresponding discrete object, while taking care to control various error terms that accumulate in the discrete setting, but previously did not appear in the continuous setting. An example of this type of argument can be found in [37]. Here we are forced to take a different path: we prove new continuous theorems, e.g. Theorems 3.1 and 7.1, which yield "rate" and "stability" versions of the previously established qualitative theorems. Once such a task is carried out, passing to the required discrete version is often quite simple.

The need to prove stability versions of certain qualitative results is a recurring theme in geometric analysis and partial differentiation equations. As a recent example one can take the stability version of the isoperimetric theorem in $\mathbb{R}^n$ that was proved in [22]. Another famous example of this type is the Sphere Theorem in Riemannian geometry (see [8] and the references therein).

In [16] we explain how our argument can be viewed as a general scheme for proving such results. The crucial point is to isolate a quantity which is *coercive, monotone over scales, and admits an a-priori bound*. In our case this quantity is the total non-monotonicity. Coercivity refers to the fact that if this quantity vanishes then a certain rigid (highly constrained) structure is enforced. Such a statement is called a *rigidity* result, and in our setting it corresponds to the classification of monotone sets in [15]. More generally (and often much harder to prove), the coercive quantity is required to have the following *almost rigidity* property: if it is less than ϵ^a, for some $a \in (0, \infty)$, then in a suitable sense, the structure is ϵ-close to the one which is forced by the $\epsilon = 0$ case. In our setting this corresponds to Theorem 7.1, and as is often the case, its proof is involved and requires insights that go beyond what is needed for the rigidity result. The monotonicity over scales refers to the decomposition (20), and the a priori bound (21), which is a consequence of the Lipschitz condition for f, implies, as in Section 9, the existence of a controlled scale at which the coercivity can be applied. We point out the general character of the estimate for the scale thus obtained, which is the reason for the logarithmic behavior in Theorem 3.1: such an estimate for the scale will appear *whenever* we are dealing with a nonnegative quantity which can be written as a sum of nonnegative terms, one controlling each scale, such that there is a definite bound on the sum of the terms. We call such a quantity *monotone over scales* to reflect the fact that the sum is nondecreasing as we include more and more scales. As one example among very many, the framework that was sketched above can be applied in the context of [13].

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